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ELASTIC CONTACT OF SMOOTH COMPLEX-SHAPED BODIES

A. A. Korolev

(Received 17 December 2001)

The contact problem for smooth elastic bodies with a single-point initial contact (the Hertz problem) was considered in numerous publications [1–6]. One of the basic assumptions under which this problem can be solved is that near the point of the initial contact, the contact surfaces are represented, as a rule, in terms of a homogeneous second degree polynomial, and therefore, the initial gap function is also described by a second degree polynomial. However, this restriction is justified only if the dimensions of the contact region are small relative to those of the contacting bodies. In practice, there are many contact problems in which the contact region is sufficiently large or the contacting bodies have a complex geometrical structure described by nonhomogeneous equations of arbitrary degree. Such problems cannot be treated in the framework of the Hertz problem. Of special interest is the contact of bodies with different shapes of the initial gap on the principal cross sections, with the equations of different degrees describing these shapes. In what follows, this case will be referred to as contact of bodies of complex geometrical structure.

Consider contact of two complex-shaped elastic bodies subjected to normal load P. Assume that there are no tangential stresses and that in the absence of normal loads, the bodies have an initial single-point contact. Let us take this point as origin O of a Cartesian coordinate system (see Figure). The axes X and Y lie in the common tangent plane, and the axis Z passes through the point O, is orthogonal the plane XOY, and is directed inside one of the contacting bodies. Suppose that the principal cross-sections of the contacting bodies belong to the same planes (for definiteness, the planes $X = 0$ and $Y = 0$).

Consider two points $M_1(x, 0, z_1)$ and $M_2(x, 0, z_2)$ of the first and the second bodies, respectively, on the plane $Y = 0$, and also points $N_1(0, y, z_1)$, $N_2(0, y, z_2)$ of the two bodies on plane the $X = 0$. Suppose that the initial distances $|M_1M_2|$ and $|N_1N_2|$ are related by

$$|M_1M_2| = z_1 - z_2 = B|x|^m, \qquad |N_1N_2| = z_1 - z_2 = A|y|^n,$$

and that the contact region arising between the two bodies has an elliptic shape.

It should be observed that the overall shape of the contacting bodies is beyond the scope of the present paper and is assumed such as to allow for the above hypotheses. Then, instead of the boundary conditions requiring information about the shape of the bodies, we have the following equations:

$$w_1(s,0,0) + w_2(x,0,0) = \delta - (z_1 - z_2) = \delta - B|x|^m, \quad w_1(0,y,0) + w_2(0,y,0) = \delta - (z_1 - z_2) = \delta - A|b|^n, \tag{1}$$

where $w_1(x, y, z)$ and $w_2(x, y, z)$ are vertical displacements of the first and the second bodies, respectively; δ is the distance by which the bodies approach one another under the action of the normal load.

On the other hand, if we denote by $q(x, y)$ the density of the single layer potential at the point (x, y), then the vertical displacement at that point can be expressed by the formula

$$w(x,y,0) = w_1(x,y,0) + w_2(x,y,0) = \left(\frac{1-\mu_1^2}{\pi E1} + \frac{1-\mu_2^2}{\pi E_2}\right) \iint_S \frac{q(x',y')\,dx'\,dy'}{\sqrt{(x-x')^2 + (y-y')^2}},$$

where S is the elliptic contact region, $x^2/a^2 + y^2/b^2 = 1$; μ_1 and μ_2 are Poisson's ratios; E_1 and E_2 are the elastic moduli of the first and the second body, respectively.

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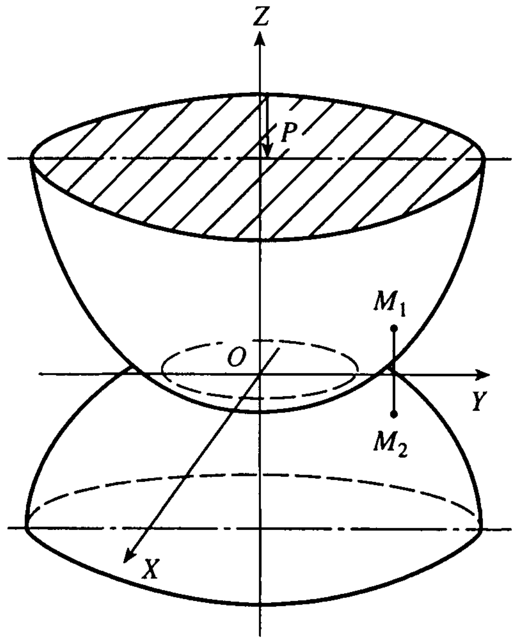

Keeping in mind the above remarks, we do not consider this integral equations in the general form and replace it by two more simple conditions. Since $|M_1M_2| = B|x|^m$, $|N_1N_2| = A|y|^n$, equations (1) become

$$\left(\frac{1-\mu_1^2}{\pi E1}+\frac{1-\mu_2^2}{\pi E_2}\right)\iint_S \frac{q(x',y')\,dx'\,dy'}{\sqrt{(x-x')^2+y^2}} = \delta - B|x|^m, \quad \left(\frac{1-\mu_1^2}{\pi E1}+\frac{1-\mu_2^2}{\pi E_2}\right)\iint_S \frac{q(x',y')\,dx'\,dy'}{\sqrt{x'^2+(y-y')^2}} = \delta - A|y|^n. \tag{2}$$

In what follows, it is assumed that the external load compressing the elastic bodies is non-instantaneous, but attains its given value P by increasing from zero on a finite time interval. Then, we can consider the contact of the elastic bodies under some intermediate normal load Q. Denote by a_Q and b_Q the minor and the major semi-axes of the elliptic contact region formed under the action of the load Q, and let $a = a_P$, $b = b_P$. As the load increases by an increment dQ, we assume that one of the contacting bodies is an elliptic punch and the other body is an elastic half-space.

As we know, the stress arising in the region of contact under the action of a plane punch subjected to normal load dQ is given by

$$dq(x,y) = \frac{dQ}{2\pi a_Q b_Q \sqrt{1-\dfrac{x^2}{a_Q^2}-\dfrac{y^2}{b_Q^2}}}, \tag{3}$$

and the distance by which the bodies approach one another is

$$d\delta_Q = \frac{\nu_0}{b_q} K(e_Q)\,dQ, \tag{4}$$

where e_a is the eccentricity of the contact ellipse arising under the external load Q; $K(e_Q)$ is an elliptic integral of the first kind,

$$\nu_0 = \frac{1-\mu_1^2}{\pi E_1} + \frac{1-\mu_2^2}{\pi E_2}.$$

We denote by (x_P, y_P) the coordinates of a point of the contact ellipse corresponding to the load P and introduce the polar coordinates

$$\xi = x_P - l\cos\varphi, \qquad \eta = y_P - l\sin\varphi,$$

where l is the polar radius of the profile of the contact region and φ is the polar angle.

Then, in view of (3), the vertical displacement at the point (x_P, y_P) caused by the load P in the contact region is given by

$$w(x_P, y_P, 0) = \nu_0 \int_0^P \int_{\varphi_0}^{\varphi_1} \int_{l_1(\varphi)}^{l_2(\varphi)} \frac{dl\, d\varphi\, dQ}{2\pi q_Q b_Q \sqrt{1 - \left(\frac{x_P - l\cos\varphi}{a_Q}\right)^2 - \left(\frac{y_P - l\sin\varphi}{b_Q}\right)^2}}, \tag{5}$$

where the limits of the interior integrals correspond to the polar boundaries of the contact region formed under the action of the intermediate normal load Q.

The limits of the interior integral, $l_1(\varphi)$ and $l_2(\varphi)$, are found from the equation

$$1 - \left(\frac{x_P - l\cos\varphi}{a_Q}\right)^2 - \left(\frac{y_P - l\sin\varphi}{b_Q}\right)^2 = 0,$$

whose roots have the form

$$l_{1,2}(\varphi) = \frac{1}{a_Q^2 \sin^2\varphi + b_Q^2\cos^2\varphi}\Big[b_Q^2 x_P\cos\varphi + a_Q^2 y_P \sin\varphi$$
$$\pm\, a_Q b_Q \sqrt{a_Q^2\sin^2\varphi + b_Q^2\cos^2\varphi - (x_P\sin\varphi - y_P\cos\varphi)^2}\Big].$$

Then, direct integration yields

$$\int_{l_1(\varphi)}^{l_2(\varphi)} \frac{dl}{2\pi a_Q b_Q \sqrt{1 - \left(\frac{x_P - l\cos\varphi}{a_Q}\right)^2 - \left(\frac{y_P - l\sin\varphi}{b_Q}\right)^2}} = \frac{1}{2b_Q\sqrt{1 - e_Q^2\sin^2\varphi}}.$$

Therefore, (5) takes the form

$$w(x_P, y_P, 0) = \frac{1}{2}\nu_0 \int_0^P \frac{1}{b_Q} \int_{\varphi_0(x_P,y_P)}^{\varphi_1(x_P,y_P)} \frac{d\varphi\, dQ}{\sqrt{1 - e_Q^2\sin^2\varphi}}.$$

Now, we can write the boundary condition (2) as follows:

$$\begin{aligned} \nu_0 \int_0^P \frac{1}{b_Q}\int_0^{\pi/2} \frac{d\varphi\, dQ}{\sqrt{1-e_Q^2\sin^2\varphi}} - \frac{1}{2}\nu_0 \int_0^P \frac{1}{b_Q}\int_{\varphi_0(a,0)}^{\varphi_1(a,0)} \frac{d\varphi\, dQ}{\sqrt{1-e_Q^2\sin^2\varphi}} &= Ba^m, \\ \nu_0 \int_0^P \frac{1}{b_Q}\int_0^{\pi/2} \frac{d\varphi\, dQ}{\sqrt{1-e_Q^2\sin^2\varphi}} - \frac{1}{2}\nu_0 \int_0^P \frac{1}{b_Q}\int_{\varphi_0(0,b)}^{\varphi_1(0,b)} \frac{d\varphi\, dQ}{\sqrt{1-e_Q^2\sin^2\varphi}} &= Ab^n, \end{aligned} \tag{6}$$

where, in view of (4),

$$\delta = \nu_0 \int_0^P \frac{1}{b_Q}\int_0^{\pi/2} \frac{d\varphi\, dQ}{\sqrt{1-e_Q^2\sin^2\varphi}}. \tag{7}$$

The limits of the interior integrals in (6) can be easily found. After simple transformations, expressions (6) become

$$\begin{aligned} &\nu_0 \int_0^P \frac{1}{b_Q}\int_0^{\pi/2-\varphi_b} \frac{d\varphi\, dQ}{\sqrt{1-e_Q^2\sin^2\varphi}} = Ab^n, \\ &\nu_0 \int_0^P \frac{1}{b_Q}\int_0^{\pi/2} \frac{d\varphi\, dQ}{\sqrt{1-e_Q^2\sin^2\varphi}} - \nu_0 \int_0^P \frac{1}{b_Q}\int_0^{\varphi_a} \frac{d\varphi\, dQ}{\sqrt{1-e_Q^2\sin^2\varphi}} = Ba^m, \end{aligned} \tag{8}$$

$$\varphi_b = \arctan\frac{a_Q}{b\sqrt{1-b_Q^2 b^{-2}}}, \qquad \varphi_a = \arctan\frac{b_Q}{a\sqrt{1-a_Q^2 a^{-2}}}. \tag{9}$$

Next, assume that the dimensions of the contact region are related to the current external load Q by

$$a_Q = uQ^h, \qquad b_Q = kQ^t, \tag{10}$$

in particular, $a = uP^h$, $b = kP^t$. Let $s = b_Q/b$. Then

$$s = \left(\frac{Q}{P}\right)^t, \qquad Q = Ps^{1/t}; \qquad dQ = P\frac{1}{t}s^{1/t-1}\,ds. \tag{11}$$

In view of (11), expressions (10) can be written as

$$a_S = uP^h s^{h/t} = as^{h/t}, \qquad b_S = kP^t s = bs. \tag{12}$$

In this case, the eccentricity of the contact ellipse corresponding to the load Q is defined by

$$e_S = \sqrt{1 - \frac{a_S^2}{b_S^2}} = \sqrt{1 - \frac{a^2}{b^2}s^{2(h/t-1)}} = \sqrt{1 - \left(1 - e^2\right)s^{2(h/t-1)}}\,,$$

where e is the eccentricity of the ellipse corresponding to the load P.

Taking into account the new notation, we can rewrite boundary conditions (8) in the form

$$\begin{aligned} &\frac{P\nu_0}{bt}\int_0^1 s^{1/t-2}\left(\int_0^{\pi/2}\frac{d\varphi\,ds}{\sqrt{1-e_S^2\sin^2\varphi}} - \int_0^{\varphi_a}\frac{d\varphi\,ds}{\sqrt{1-e_S^2\sin^2\varphi}}\right) = Ba^m, \\ &\frac{P\nu_0}{bt}\int_0^1 s^{1/t-2}\int_0^{\pi/2-\varphi_b}\frac{d\varphi\,ds}{\sqrt{1-e_S^2\sin^2\varphi}} = Ab^n. \end{aligned} \tag{13}$$

Denote the integrals in (13) by

$$\begin{aligned} J_0(e) &= \int_0^1 s^{n-1}\int_0^{\pi/2}\frac{d\varphi\,ds}{\sqrt{1-e_S^2\sin^2\varphi}}, \\ J_a(e) &= \int_0^1 s^{n-1}\int_0^{\varphi_a}\frac{d\varphi\,ds}{\sqrt{1-e_S^2\sin^2\varphi}}, \\ J_b(e) &= \int_0^1 s^{n-1}\int_0^{\pi/2-\varphi_b}\frac{d\varphi\,ds}{\sqrt{1-e_S^2\sin^2\varphi}}. \end{aligned} \tag{14}$$

Then, it follows from the second relation in (13) that

$$P^{1/(n+1)}\left(\frac{\nu_0}{At}J_b(e)\right)^{1/(n+1)} = b.$$

On the other hand, from (12) we see that $b = kP^t$. Comparing the exponents of P in the last two relations, we find that $t = 1/(n+1)$. From the first expression in (13), keeping in mind the new notation, we find that

$$a = P^{n/[m(n+1)]}\left(\frac{\nu_0(J_0(e) - J_a(e))A^{1/(n+1)}(n+1)}{[\nu_0(n+1)J_b(e)]^{B/(n+1)}}\right)^{1/m}.$$

By (12), we have $a = uP^h$, and therefore, comparing the last two expressions, we obtain for the exponents of P

$$h = \frac{n}{m(n+1)}.$$

Therefore, relations (9) take the form

$$\varphi_a = \arctan\left(\frac{s}{\sqrt{1-e^2}\,\sqrt{1-s^{2n/m}}}\right), \qquad \varphi_b = \arctan\left(\frac{\sqrt{1-e^2}\,s^{n/m}}{\sqrt{1-s^2}}\right). \tag{15}$$

The eccentricity of the contact ellipse corresponding to the intermediate load is equal to

$$e_S = \sqrt{1 - \left(1 - e^2\right) s^{2(n/m-1)}} . \tag{16}$$

This expression shows that the eccentricity of the contact ellipse is constant only if $n = m$. For $n \neq m$, the eccentricity depends on the external load.

With reference to (15) and (16) for $n = m$, the integrals (14) can be simplified. Thus,

$$J_0(e) = \int_0^1 s^{n-1} \int_0^{\pi/2} \frac{d\varphi\, ds}{\sqrt{1 - e^2 \sin^2 \varphi}} = K(e) \frac{s^n}{n} \bigg|_0^1 = \frac{1}{n} K(e).$$

Let us simplify $J_a(e)$ by integrating by parts. Let

$$u(s) = \int_0^{\alpha(s)} \frac{d\varphi}{\sqrt{1 - e^2 \sin^2 \varphi}}, \qquad \alpha(s) = \arctan \frac{s}{\sqrt{1 - s^2}\,\sqrt{1 - e^2}}, \qquad d\nu = s^{n-1}\, ds.$$

Then,

$$u'(s) = \left(\arctan \frac{s}{\sqrt{1 - s^2}\,\sqrt{1 - e^2}} \right)' \frac{1}{\sqrt{1 - e^2 \sin^2 \left(\arctan \frac{s}{\sqrt{1 - s^2}\,\sqrt{1 - e^2}} \right)}}$$

$$= \frac{1}{\sqrt{1 - s^2}\,\sqrt{1 - e^2 + e^2 s^2}}, \qquad \nu = \frac{s^n}{n}.$$

Hence,

$$J_a(e) = \int_0^1 s^{n-1} \int_0^{\varphi_a} \frac{d\varphi\, ds}{\sqrt{1 - e_s^2 \sin^2 \varphi}}$$

$$= \frac{s^n}{n} \int_0^{\alpha(s)} \frac{d\varphi}{\sqrt{1 - e^2 \sin^2 \varphi}} \bigg|_0^1 - \frac{1}{n} \int_0^1 \frac{s^n\, ds}{\sqrt{1 - s^2}\,\sqrt{1 - e^2 - e^2 s^2}} .$$

Changing the variables, $s = \cos \varphi$, we get

$$J_a(e) = \frac{1}{n} K(e) - \frac{1}{n} \int_0^{\pi/2} \frac{\cos^n \varphi\, d\varphi}{\sqrt{1 - e^2 \sin^2 \varphi}} .$$

In a similar way, we simplify the expression of the integral $J_b(e)$. Thus, for $n = m$, expressions (14) are transformed to

$$\begin{aligned} J_0(e) &= \frac{1}{n} K(e), \\ J_a(e) &= \frac{1}{n} K(e) - \frac{1}{n} \int_0^{\pi/2} \frac{\cos^n \varphi\, d\varphi}{\sqrt{1 - e^2 \sin^2 \varphi}}, \\ J_b(e) &= \frac{1}{n} \int_0^{\pi/2} \frac{\sin^n \varphi\, d\varphi}{\sqrt{1 - e^2 \sin^2 \varphi}} . \end{aligned} \tag{17}$$

The formulas expressing the dimensions of the contact region can be written as follows:

$$\begin{aligned} a &= P^{n[m(n+1)]} \left(\frac{\nu_0 (J_0(e) - J_a(e)) A^{1/(n+1)} (n+1)}{[\nu_0 (n+1) J_b(e)]^{1/(n+1)} B} \right)^{1/m}, \\ b &= P^{1/(n+1)} \left(\frac{\nu_0 (n+1)}{A} J_b(e) \right)^{1/(n+1)} . \end{aligned} \tag{18}$$

Dividing the first relation in (18) by the second relation, we obtain

$$\frac{a}{b} = P^{\frac{n-m}{m(n+1)}} [\nu_0 (n+1)]^{\frac{n-m}{m(n+1)}} \frac{A^{\frac{m+1}{m(n+1)}} (J_0(e) - J_a(e))^{\frac{1}{m}}}{B^{\frac{1}{m}} J_b^{\frac{m+1}{m(n+1)}}(e)} .$$

On the other hand, we know that

$$\frac{a}{b} = \sqrt{1-e^2}.$$

Equating the right-hand sides of the last two relations, we obtain the equation for the eccentricity,

$$\sqrt{1-e^2} = P^{\frac{n-m}{m(n+1)}} [\nu_0(n+1)]^{\frac{n-m}{m(n+1)}} \frac{A^{\frac{n-m}{m(n+1)}}}{B^{\frac{1}{m}}} \frac{(J_0(e) - J_a(e))^{\frac{1}{m}}}{J_b^{\frac{m+1}{m(n+1)}}(e)}. \tag{19}$$

Having found the eccentricity from equation (19), we use (18) to find the length of the major semi-axis of the contact ellipse, b, and then we find the length of its minor semi-axis, a.

Obviously, relation (19) imposes a constraint on the coefficients A and B. Since the left-hand side of (19) can take values from 0 to 1, we see that for given P, n, m, and ν_0, the coefficients A and B should be such that the right-hand side of (19) would also range from 0 to 1.

In particular, in order to find A and B for which equation (19) has a solution with respect to e for $n = m$, we rewrite this equation, taking into account (17), as follows:

$$\left(1-e^2\right)^{\pi/2} \int_0^{\pi/2} \frac{\sin^n \varphi \, d\varphi}{\sqrt{1-e^2\sin^2\varphi}} \left[\int_0^{\pi/2} \frac{\cos^n \varphi \, d\varphi}{\sqrt{1-e^2\sin^2\varphi}}\right]^{-1} = \frac{A}{B}. \tag{20}$$

If $0 \le e \le 1$, then the left-hand side of the last expression ranges from 1 to 0. Therefore, the solution of equation (20) exists if $0 \le A/B \le 1$.

Let us compare the results obtained above with the Hertz formulas for $n = m = 2$. From the second relation of (18) it follows that

$$b = \sqrt[3]{P\frac{3\nu_0}{2A} 2J_b(e)}.$$

From the last relation of (17), we obtain

$$J_b(e) = \frac{1}{2}\int_0^{\pi/2} \frac{\sin^2\varphi \, d\varphi}{\sqrt{1-e^2\sin^2\varphi}} = \frac{1}{2}D(e) = \frac{1}{2e^2}[K(e) - E(e)],$$

where $K(e)$ and $E(e)$ are the elliptic integrals of the first and the second kind, respectively. Therefore, the length of major semi-axis of the contact region is

$$b = \sqrt[3]{P\frac{3\nu_0}{2A} D(e)},$$

which coincides with the Hertz formula.

Let us find the distance δ by which the contacting elastic bodies approach one another under the normal load P. Substituting (7) into (12), we obtain

$$\delta = \nu_0 \frac{P(n+1)}{b} \int_0^1 s^{n-1} \int_0^{\pi/2} \frac{d\varphi}{\sqrt{1 - \left(1 - \left(1-e^2\right) s^{2n/(m-1)}\right)\sin^2\varphi}} \, ds.$$

In particular, for $n = m$, we have

$$\delta = \nu_0 \frac{3P}{b} \int_0^1 s^{n-1} \int_0^{\pi/2} \frac{d\varphi}{\sqrt{1-e^2\sin^2\varphi}} \, ds = \nu_0 \frac{3P}{nb} K(e),$$

which coincides with the Hertz formulas for $n = 2$ and with the I. Ya. Shtaerman formula for $n = 4$.

Let us find the distribution of contact stresses. It is known that the equilibrium condition in the case of contact under consideration can be written as

$$P = \iint_S q(x,y)\, dS,$$

where S is the elliptic contact region, and $q(x,y)$ is the stress arising at the point (x,y) in the contact region under the action of the external load P.

Denote by P_{xy} the external load for which (x, y) becomes a boundary point of the contact region. Obviously, as the load increases, (x, y) becomes an interior point of the contact region. Then, under the external load P, the stress arising at the point (x, y) can be calculated by the integration of (3),

$$q(x,y) = \frac{1}{2\pi} \int_{P_{xy}}^{P} \frac{dQ}{a_Q b_Q \sqrt{1 - \dfrac{x^2}{a_Q^2} - \dfrac{y^2}{b_Q^2}}} .$$

Using (11) and (12), we can rewrite the last expression as

$$\begin{aligned} q(x,y) &= \frac{P(n+1)}{2\pi ab} \int_{s_0}^{1} \frac{s^{n-n/m-1}\,ds}{\sqrt{1 - \dfrac{x^2}{a^2} s^{2n/m} - \dfrac{y^2}{b^2} s^2}} \\ &= \frac{P(n+1)}{2\pi ab} \int_{s_0}^{1} \frac{s^{n-1}\,ds}{\sqrt{s^{2n/m} - \dfrac{y^2}{b^2} s^{2n/m-2} - \dfrac{x^2}{a^2}}} , \end{aligned} \tag{22}$$

where $s_0 = P_{xy}/P$ is the positive real root of the following equation with respect to s:

$$s^{2n/m} - \frac{y^2}{b^2} s^{2n/m-2} - \frac{x^2}{a^2} = 0. \tag{23}$$

Relations (22) and (23) yield the final solution of the problem formulated above.

In order to transform (22), we introduce the variables

$$t = \frac{x}{a}, \qquad p = \frac{y}{b} .$$

Then (22) becomes

$$q_1(t,p) = q(at, bp) = \frac{P(n+1)}{2\pi ab} \int_{s_1(t,p)}^{1} \frac{s^{n-1}\,ds}{\sqrt{s^{2n/m} - p^2 s^{2n/m-2} - t^2}} , \tag{25}$$

where $s_1(r,p)$ is a real root of the equation

$$s^{2n/m} - p^2 s^{2n/m-2} - t^2 = 0. \tag{26}$$

Thereby, we have reduced the problem of finding contact stresses on the elliptic contact region to that of finding stresses on a contact region in the form of a unit circle. Obviously, making the transformation inverse to that of (24), we obtain the stresses at the point of the elliptic contact region.

In particular, from (2.5) it follows that the stress at the center of the contact region, $p = 0,\ t = 0$, is equal to

$$p_0 = q_1(0,0) = \frac{P(n+1)}{2\pi ab} \int_0^1 s^{n-1-n/m}\,ds = \frac{P}{2\pi ab} \frac{m(n+1)}{n(m-1)} . \tag{27}$$

It is easy to see that for $n = m = 2$ we obtain an expression which coincides with the Hertz formula.

Passing from the stress function on the elliptic contact region to the stress function on the unit circle contact region, we can rewrite the equilibrium condition (21) in the form

$$\iint_S q(x,y)\,dS = 4 \int_a^b \int_0^{a\sqrt{1-y^2/b^2}} q(x,y)\,dx\,dy = 4ab \int_0^1 \int_0^{\sqrt{1-p^2}} q_1(t,p)\,dt\,dp = ab \iint_{S_1} q_1(t,p) = P, \tag{28}$$

where S_1 is the unit circle.

For arbitrary n and m, the stresses can be easily found from (25) and (26) by numerical methods. However, in some special cases, these expressions can be substantially simplified.

Let $m = n$. Then (25) becomes

$$q_1(t,p) = \frac{P(n+1)}{2\pi ab} \int_{\sqrt{p^2+t^2}}^{1} \frac{s^{n-1}\,ds}{\sqrt{s^2-p^2-t^2}}\,. \tag{29}$$

This integral can be expressed in terms of a hypergeometric function of the first kind. To that end, we change the variables by letting

$$z = \sqrt{s^2-p^2-t^2}\,, \qquad s = \sqrt{z^2+p^2+t^2}\,, \qquad ds = \frac{t\,dt}{\sqrt{z^2+p^2+t^2}}\,.$$

Then

$$\begin{aligned} q_1(t,p) &= \frac{P(n+1)}{2\pi ab} \int_0^{\sqrt{1-p^2-t^2}} \left(t^2+p^2+t^2\right)^{(n-2)/2} dt \\ &= \frac{P(n+1)}{2\pi ab} \sqrt{1-p^2-t^2}\left(p^2+t^2\right)^{(n-2)/2} {}_2F_1\left(\frac{1}{2}\,,\ 1-\frac{n}{2}\,,\ \frac{3}{2}\,,\ -\frac{1-p^2-t^2}{p^2+t^2}\right) \end{aligned}$$

where ${}_2F_1\left(\frac{1}{2}\,,\ 1-\frac{n}{2}\,,\ \frac{3}{2}\,,\ -\frac{1-p^2-t^2}{p^2+t^2}\right)$ is a hypergeometric function of the first kind.

In particular, for $n = 2$, we have

$${}_2F_1\left(\frac{1}{2}\,,\ 0\,,\ \frac{3}{2}\,,\ -\frac{1-p^2-t^2}{p^2+t^2}\right) = 1,$$

and therefore,

$$q_1(t,p) = \frac{P(n+1)}{2\pi ab}\sqrt{1-p^2-t^2}\,,$$

or

$$q(x,y) = \frac{3P}{2\pi ab}\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}\,,$$

which agrees with the corresponding Hertz formula.

For $n = 4$, we have

$${}_2F_1\left(\frac{1}{2}\,,\ -1\,,\ \frac{3}{2}\,,\ -\frac{1-p^2-t^2}{p^2+t^2}\right) = 1 + \frac{1}{3}\,\frac{1-p^2-t^2}{p^2+t^2}\,.$$

It follows that

$$\begin{aligned} q_1(t,p) &= \frac{5P}{2\pi ab}\sqrt{1-p^2-t^2}\,\left(p^2+t^2\right)\left(1+\frac{1}{3}\,\frac{1-p^2-t^2}{p^2+t^2}\right) \\ &= \frac{5P}{2\pi ab}\sqrt{1-p^2-t^2}\left(p^2+t^2+\frac{p^2}{3}-\frac{t^2}{3}\right) = \frac{5P}{2\pi ab}\sqrt{1-p^2-t^2}\left(1-\frac{2}{3}\left(1-p^2-t^2\right)\right), \end{aligned}$$

or

$$q(x,y) = \frac{5P}{2\pi ab}\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}\left(1-\frac{2}{3}\left(1-\frac{x^2}{a^2}-\frac{y^2}{b^2}\right)\right),$$

which coincides with the expression obtained by I. Ya. Shtaerman [4].

Changing the variable by $\nu = \sqrt{p^2+t^2}\,/s$, we can represent (29) in another form,

$$q_1(t,p) = \frac{P(n+1)}{2\pi ab}\left(p^2+t^2\right)^{(n-1)/2} \int_{\sqrt{p^2+t^2}}^{1} \frac{d\nu}{\nu^n\sqrt{1-\nu^2}}\,. \tag{30}$$

This formula will be used for the verification of the equilibrium condition.

Let us introduce the polar coordinates $p = r\cos\varphi$, $t = r\sin\varphi$. Then, (30) can be written as

$$q_2(r) = \frac{P(n+1)}{2\pi ab} r^{n-1} \int_r^1 \frac{d\nu}{\nu^n\sqrt{1-\nu^2}}\,. \tag{31}$$

Table

n	$N_n(r)$
1	2
2	1
3	$1+\dfrac{r^2}{\sqrt{1-r^2}}\operatorname{Artanh}\sqrt{1-r^2}$
4	$1+2r^2$
5	$1+\dfrac{3}{2}r^2+\dfrac{3}{2}\dfrac{r^4}{\sqrt{1-r^2}}\operatorname{Artanh}\sqrt{1-r^2}$
6	$1+\dfrac{4}{3}r^2+\dfrac{8}{3}r^4$
7	$1+\dfrac{5}{4}r^2+\dfrac{15}{8}\dfrac{r^6}{\sqrt{1-r^2}}\operatorname{Artanh}\sqrt{1-r^2}$
8	$1+\dfrac{6}{5}r^2+\dfrac{8}{5}r^4+\dfrac{16}{5}r^6$

Condition (28) takes the form

$$ab\iint_S q_2(r)\,dS = ab\int_0^1 l(r)q_2(r)\,dr = P,$$

where $l(r) = 2\pi r$ is the length of a circle of radius r.

To prove the last relation, we substitute (31) into its left-hand side. We get

$$ab\int_0^1 l(r)q_2(r)\,dr = ab\int_0^1 2\pi r q_2(r) = P(n+1)\int_0^1 r^n \int_r^1 \frac{d\nu}{\nu^n\sqrt{1-\nu^2}}\,dr.$$

Let us integrate by parts, using the notation

$$u = \int_r^1 \frac{d\nu}{\nu^n\sqrt{1-\nu^2}}, \qquad d\nu = r^n\,dr, \qquad du = -\frac{dr}{r^n\sqrt{1-r^2}}, \qquad \nu = \frac{r^{n+1}}{n+1}.$$

We find that

$$\begin{aligned} ab\int_0^1 2\pi r q_2(r)\,dr &= P(n+1)\int_0^1 r^n\int_r^1 \frac{d\nu}{\nu^n\sqrt{1-\nu^2}}\,dr \\ &= P(n+1)\left(\frac{r^{n+1}}{n+1}\int_r^1 \frac{d\nu}{\nu^n\sqrt{1-\nu^2}}\bigg|_0^1 - \int_0^1 \frac{r^{n+1}}{n+1}\left(-\frac{1}{r^n\sqrt{1-r^2}}\right)dr\right) \\ &= P(n+1)\int_0^1 \frac{r^{n+1}}{n+1}\left(\frac{1}{r^n\sqrt{1-r^2}}\right)dr = P\int_0^1 \frac{r\,dr}{\sqrt{1-r^2}} = P. \end{aligned}$$

Therefore, the equilibrium condition holds for the stress functions (29), (30), or (31) for arbitrary $n = m$.

The stresses in the case of $n = m$ were first calculated in [6] without the verification of the equilibrium condition. In order to compare the formulas obtained for the stresses in the present paper with those of [6], we represent (31) as follows:

$$q(r) = p_0(n-1)r^{n-1}\int_r^1 \frac{d\nu}{\nu^n\sqrt{1-\nu^2}}.$$

Letting

$$N_n(r) = \frac{r^{n-1}(n-1)}{\sqrt{1-r^2}}\int_r^1 \frac{d\nu}{\nu^n\sqrt{1-\nu}}$$

and substituting this into (31), we obtain the formula of [6],

$$q(x,y) = p_0\sqrt{1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}}\, N_n(r), \qquad r = \sqrt{\frac{x^2}{a^2} + \frac{y^2}{b^2}}\,.$$

The expressions of $N_n(r)$ for various n presented in the table.

There are several more special cases in which (25), (26) can be simplified. Consider the cases $m = 2$, $n = 4$; $m = 3$, $n = 6$; and $m = 4$, $n = 8$.

In all these cases, the lower integration limit in (25) coincides with a root of the equation $s^4 - p^2s^2 - t^2 = 0$. This root is

$$s_1 = \sqrt{\frac{p^2}{2} + \frac{1}{2}\sqrt{p^4 + 4t^2}}\,.$$

Let $m = 2$, $n = 4$. Then (25) takes the form

$$q_1(t,p) = \frac{5P}{2\pi ab}\int_{s_1(t,p)}^{1} \frac{s^3\,ds}{\sqrt{s^4 - p^2s^2 - t^2}}\,.$$

Calculating the last integral, we obtain

$$q_1(t,p) = \frac{5P}{2\pi ab}\int_{\sqrt{2^{-1}p^2+2^{-1}\sqrt{p^4+4t^2}}}^{1} \frac{s^3\,ds}{\sqrt{s^4 - p^2s^2 - t^2}}$$

$$= \frac{5P}{2\pi ab}\frac{1}{4}\left(2\sqrt{1-p^2-t^2} + p^2\ln\left(\frac{2-p^2+2\sqrt{1-p^2-t^2}}{\sqrt{p^2+4t^2}}\right)\right). \qquad (32)$$

Let us verify the equilibrium condition. We have

$$ab\iint_{S_1} q_1(t,p)\,dS_1 = 4\frac{5P}{2\pi}\frac{1}{4}\int_0^1\int_0^{\sqrt{1-p^2}} 2\sqrt{1-p^2-t^2}\,dt\,dp$$

$$+ 4\frac{5P}{2\pi}\frac{1}{4}\int_0^1\int_0^{\sqrt{1-p^2}} p^2\ln\left(\frac{2-p^2+2\sqrt{1-p^2-t^2}}{\sqrt{p^2+4t^2}}\right)dt\,dp = \frac{5P}{\pi}\frac{\pi}{6} + \frac{5P}{2\pi}\frac{\pi}{15} = P,$$

and therefore, the equilibrium condition holds.

Let $m = 3$, $n = 6$. Then (25) becomes

$$q_1(t,p) = \frac{7P}{2\pi ab}\int_{s_1(t,p)}^{1} \frac{s^5\,ds}{\sqrt{s^4 - p^2s^2 - t^2}}\,.$$

Hence,

$$q_1(t,p) = \frac{7P}{2\pi ab}\frac{1}{16}\Big\{(4+6p^2)\sqrt{1-p^2-t^2} - (3p^4+4t^2)\ln\sqrt{p^2+4t^2}$$

$$+ (3p^4+4t^2)\ln\left(2-p^2+2\sqrt{1-p^2-t^2}\right)\Big\}. \qquad (33)$$

The equilibrium condition holds, since

$$ab\int_{S_1} q_1(t,p)\,dS_1 = 4\frac{7P}{2\pi}\frac{1}{16}\int_0^1\int_0^{\sqrt{1-p^2}} (4+6p^2)\sqrt{1-t^2-p^2}\,dt\,dp$$

$$-4\frac{7P}{2\pi}\frac{1}{16}\int_0^1\int_0^{\sqrt{1-p^2}} (3p^4+4t^2)\ln\sqrt{p^2+4t^2}\,dt\,dp$$

$$+4\frac{7P}{2\pi}\frac{1}{16}\int_0^1\int_0^{\sqrt{1-p^2}} (3p^4+4t^2)\ln\left(2-p^2+2\sqrt{1-p^2-t^2}\right)dt\,dp = 4\frac{7P}{2\pi}\frac{\pi}{14} = P.$$

For $n = 8$, $m = 4$, we have

$$q_1(t,p) = \frac{9P}{2\pi ab} \int_{s_1(t,p)}^{1} \frac{s^7\,ds}{\sqrt{s^4 - p^2 s^2 - t^2}},$$

and therefore,

$$q_1(t,p) = \frac{9P}{2\pi ab}\frac{1}{192}\Big\{4\sqrt{1-p^2-t^2}\left(8 + 10p^2 + 15p^4 + 16t^2\right) - 3\left(5p^6 + 12p^2t^2\right)\Big(\ln\left(p^4 + 4t^2\right) - 2\ln\Big(2 - p^2 + 2\sqrt{1-p^2-t^2}\Big)\Big)\Big\}. \qquad (34)$$

Simple calculations show that this expression also satisfies the equilibrium condition.

It should be noted that in most cases, such expressions for the stress function may be very cumbersome. At the same time, for arbitrary parameters n and m of the gap function, formulas (25) and (26) yield very good results for the stresses in the contact region, if one resorts to numerical methods, for instance, those provided by Mathematica 4 software.

Thus, together with formulas (25), (26) for the calculation of contact stresses, we have obtained, as special cases, Hertz, Shtaerman [4], and Korolev [6] formulas. For arbitrary n and m, we have found the stresses at the center of the contact region, and obtained formulas (32), (33), and (34) for the calculation of contact stresses for $n = 4$ and $m = 2$, $n = 6$ and $m = 3$, $n = 8$ and $m = 4$.

The solution of this contact problem is of great practical importance. Thus, when manufacturing ball bearings, one can vary the gap between the balls and the races to ensure a more uniform distribution of contact stresses in the contact region, thereby increasing the life-time of the device and its bearing capacity. On the other hand, decreasing the dimensions of the contact region between the balls and the race, it is possible to increase the speed of the bearing. This statement pertains to all other coupling devices such as serrated joints, spline connections or journal bearings.

Of special practical importance are the cases with the exponent of the initial gap function along one of the principal axes being $m = 2$, and along the other principal axis, $n > 2$. For example, this is the case if all bodies and races in the rolling plane have a spherical shape and the contact region is small. Then, in the direction of rolling, the exponent of the gap function is always close to 2. However, in the transverse direction, the profile of the bodies may be taken arbitrary, so as to ensure a favorable distribution of contact stresses and thus increase the efficiency of rolling-contact bearings.

The results obtained in this paper indicate how to adjust the initial gap between contacting bodies in order to obtain optimal stress diagrams in the contact region and increase the efficiency of a wide class of mechanical devices.

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